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BOUNDARY LAYERS WITH PRESCRIBED HEAT FLUX—APPLICATION TO SIMULTANEOUS CONVECTION AND RADIATION

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Abstract—An analysis is carried out to determine the distribution of surface temperature along a flat plate experiencing simultaneous convective heat transfer, radiative exchange with the environment, aerodynamic heating, and internal heat sources or sinks. Both laminar and turbulent boundary layer flows are considered. Numerical results are presented for a wide range of the governing parameters; these are compared with simplified solutions based on local application of heat-transfer coefficients for uniform wall temperature and for uniform heat flux.

The problem is treated within the framework of the thermal boundary layer with prescribed heat flux. The initial part of the paper is devoted to establishing certain general results for such boundary layers. Exact solutions are obtained for power-law and series heat flux distributions. An approximate solution for arbitrarily varying surface heat flux is derived by superposing step-function solutions furnished by the integral energy equation.

NOMENCLATURE

a, b, constants;
C, reciprocal of $n = 0$ entries of Table 1;
c_p, specific heat, constant pressure;
e_{RAD}, radiant energy absorbed by plate/time–area;
e_p, internal heat load/time-area;
e, sum of e_{RAD} and e_p;
f, dimensionless stream function;
h, heat-transfer coefficient;
h_{RAD}, radiation coefficient, equation (27);
h_{UHF}, convective coefficient for uniform heat flux;
h_{UWT}, convective coefficient for uniform wall temperature;
k, thermal conductivity;
Nu_x, Nusselt number, hx/k;
Pr, Prandtl number, $c_p\mu/k$;
q, convective heat flux/time-area;
R, recovery factor, equation (21);
Re_x, Reynolds number, $U_\infty x/\nu$;
T, temperature;
T_∞, free steam temperature;
U_∞, free stream velocity;
u, streamwise velocity component;
v, transverse velocity component;
X, dimensionless coordinate, equation (26);
x, streamwise coordinate;
x_0, coordinate specifying heat flux distribution;
y, transverse coordinate.

Greek symbols
α, absorptance;
ϵ, emittance;
η, similarity variable, equation (2);
θ, dimensionless temperature, T_w/T_{aw};
μ, absolute viscosity;
ξ, dummy coordinate;
ρ, density;
σ, Stefan–Boltzmann constant;
ϕ, dimensionless temperature, equation (3b);
χ, dimensionless coordinate, equation (33).

Subscripts
aw, adiabatic wall condition;
w, at the surface;
∞, in the free stream.

INTRODUCTION

THERE has been considerable recent interest in boundary layer heat transfer under conditions of non-uniform thermal conditions at the surface. Such conditions may arise naturally in situations where several heat-transfer processes occur simultaneously, for instance, when the distribution of surface temperature results from the combined action of radiation, convection with or without aerodynamic heating, and heat addition or removal at the surface. This paper is concerned with the aforementioned heat-transfer problem for both cases wherein the boundary-layer flow is laminar or is turbulent.

Specific consideration is given here to flow over a flat plate which exchanges heat both by convection with the flowing fluid and by radiation with the environment (e.g. solar source, earth's albedo, and so forth). The emissivity of the plate surface may be different from its absorptivity. There may be aerodynamic heating in the boundary layer and heat addition or removal at the plate surface.

In carrying out the analysis, it is convenient to treat the problem within the framework of the thermal boundary-layer with prescribed surface heat flux. Correspondingly, the first part of the paper is devoted to establishing some general results for boundary-layer flows with prescribed heat flux. With these results in hand, consideration is then given to the problem of simultaneous radiation and convection.

Previous contributions to the radiative-convective boundary layer are due to Lighthill [1] and to Cess [2]. Lighthill limited his considerations to determining the adiabatic wall temperature distribution on an aerodynamically heated plate that was cooled by radiation at the plate surface; the boundary-layer flow was laminar. The problem was analysed by applying a superposition integral that was based on an approximate solution of the boundary-layer energy equation for prescribed surface temperature. Numerical results were obtained for the limits of weak radiative effects and of strong radiative effects, and a curve was faired in the intermediate region. The analysis of Cess was aimed at determining the first-order corrections to the convective Nusselt number due to radiative exchange between the plate and the environment; both laminar and turbulent flows were considered. The boundary condition of uniform surface heat flux was imposed; aerodynamic heating was not included. The analysis took the form of a perturbation of the convective energy equation. The results of both Lighthill and Cess will be brought together with those of the present analysis in later sections of the paper.

THERMAL BOUNDARY LAYER WITH PRESCRIBED HEAT FLUX

The heat-transfer characteristics of forced-convection boundary-layer flows are typically solved for under the condition of prescribed surface temperature. Among all boundary-layer flows, the most extensive treatment has been accorded the flat plate. Numerous solutions for both laminar and turbulent flow exist for the case of the isothermal plate. For the case of prescribed non-uniform surface temperature, two classes of solutions exist for the laminar boundary layer. The first of these includes exact similarity solutions for temperature variations having the specific form:

$$T_w - T_\infty = ax^n \quad \text{or} \quad T_w - T_\infty = \sum a_n x^n;$$

for instance, references [3] and [4]. The second accommodates any distribution of surface temperature as input to a superposition integral, the latter having been constructed from approximate solutions for a step change in temperature applied downstream of the hydrodynamic leading edge; for instance, references [5, 6, 7, and 1]. For the turbulent boundary layer, treatment of the non-isothermal case has been confined to the superposition formulation [8, 9].

It is the aim of this section of the paper to provide results for the prescribed heat flux case which complement those just cited for the case of prescribed surface temperature. Consideration will be given first to laminar flow over a flat plate, after which the turbulent case will be considered. The analysis that follows is initially concerned with constant-property, non-dissipative flows. Later, the results are modified to include the effects of viscous dissipation. Fluid property variations may be accounted for by evaluating the results at a suitable reference temperature.

Power-law and series heat flux distributions, laminar

Exact similarity solutions of the boundary-layer energy equation can be found for power-law heat flux distributions of the form

$$q = bx^n \tag{1}$$

wherein b is a constant and n may be integral or non-integral. The starting point of the analysis is the boundary-layer energy equation for constant-property, non-dissipative flow over a flat plate

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}. \tag{2}$$

If one introduces the Blasius variables

$$\eta = \frac{y}{2x} Re_x^{1/2}, \quad \frac{u}{U_\infty} = \tfrac{1}{2}f',$$

$$\frac{v}{U_\infty} = \frac{Re_x^{-1/2}}{2}(f - \eta f') \tag{3a}$$

and additionally defines a temperature similarity variable ϕ as

$$\phi(\eta) = (T - T_\infty)\Big/\frac{2qx}{k} Re_x^{-1/2} \tag{3b}$$

then, there is obtained

$$\phi'' = Pr[(2n + 1)f'\phi - f\phi']. \tag{4}$$

The boundary conditions require that $\partial T/\partial y = -q/k$ at the plate surface and that $T \to T_\infty$ in the free stream. In terms of the new variables, the boundary conditions become

$$\phi(0) = -1, \quad \phi(\infty) = 0. \tag{5}$$

The result of greatest interest here is the surface temperature distribution corresponding to the prescribed heat flux. From equation (3b), one finds

$$T_w - T_\infty = \frac{2qx}{k} Re_x^{-1/2}\phi(0) \sim x^{n+1/2}. \tag{6}$$

It is thus seen that for a surface heat flux varying as x^n, the wall temperature varies as $x^{n+1/2}$. In addition, numerical results for the surface temperature variation depend on the quantity $\phi(0)$.

Solutions of equation (4) subject to the boundary conditions (5) have been carried out numerically for Prandtl numbers of 0·7, 1, 10, and 100 for $n = 0$, 1, 2, 4, 6, and 10. The $\phi(0)$ values corresponding to these cases have been utilized in computing a Nusselt number Nu defined as follows

$$h = q/(T_w - T_\infty), \quad Nu_x = hx/k \tag{7}$$

Taking cognizance of equation (6), one finds

$$\frac{Nu_x}{Re_x^{1/2}Pr^{1/3}} = \frac{1}{2Pr^{1/3}\phi(0)} \tag{8}$$

where the factor $Pr^{1/3}$ has been included to essentially eliminate the effect of Prandtl number. The dimensionless grouping appearing on the left of equation (8) is listed in Table 1.

Table 1. $Nu_x/Re_x^{1/2}Pr^{1/3}$ corresponding to $q \sim x^n$. Exact boundary layer solutions

	Pr			
n	0·7	1	10	100
0	0·45716	0·45897	0·46318	0·46363
1	0·60544	0·60849	0·60938	0·60968
2	0·70467	0·70562	0·70766	0·70791
4	0·84653	0·84724	0·84868	0·84890
6	0·95238	0·95293	0·95411	0·95420
10	1·1129	1·1133	1·1141	1·1142

Upon inspecting the table, one sees very little effect of Prandtl number. In addition, it is seen that the Nusselt number increases with increasing n. This is a direct consequence of a thinning of the thermal boundary layer. The Nusselt number results of Table 1 can be correlated within 2 per cent by the following simple relationship

$$\frac{Nu_x}{Re_x^{1/2}Pr^{1/3}} \cong \frac{(n + 1)^{3/8}}{C}. \tag{9}$$

The constant C is the reciprocal of the entries in the $n = 0$ row of the table.

Although the present formulation is intended to provide the surface temperature corresponding to a prescribed heat flux $q \sim x^n$, the results may also be used to determine the surface heat flux corresponding to a prescribed temperature variation $(T_w - T_\infty) \sim x^{n+1/2}$. Thus, the present solutions supplement existing solutions for prescribed surface temperature.

Consideration may next be given to the case wherein the surface heat flux varies as

$$q = \sum b_n x^n \tag{10}$$

in which the exponents n are not restricted to be integers. Corresponding to the variation specified by equation (10), one may propose a temperature solution of the form

$$T - T_\infty = \frac{2x}{k} Re_x^{-1/2} \sum b_n \phi_n(\eta) x^n,$$

$$T_w - T_\infty = \frac{x}{k} Re_x^{-1/2} Pr^{-1/3} \sum b_n [2Pr^{1/3} \phi_n(0)] x^n \tag{11}$$

By making use of the linearity of the energy equation, it can be shown that ϕ_n is governed by equations (4) and (5). Therefore, the solutions for the power-law heat flux, equation (1), can be employed for the series heat flux distribution, equation (10). Correspondingly, the bracketed quantity in equation (11) is equal to the reciprocal of the numbers listed in Table 1.

Arbitrary heat flux distribution, laminar

The foregoing analysis provides exact solutions corresponding to heat flux distributions that can be represented either in the form of equations (10) or (1). To accommodate completely arbitrary heat flux distributions, it is necessary to pass to a superposition-integral type of solution. To construct such a solution, one must first obtain results for the case of a step change in surface heat flux applied at a position downstream of the hydrodynamic leading edge.

Specifically, consideration is given to a plate which is unheated in the region $0 \leqslant x < x_0$ and is uniformly heated for all $x > x_0$. It is not possible to derive an exact solution corresponding to this heating condition. However, an approximate solution may be obtained by applying the integral form of the boundary-layer energy equation. Indeed, such an analysis is contained in the text by Eckert and Drake [5] for the case of a step-change in wall temperature at x_o. That analysis may be utilized for the step-change in heat flux by replacing their θ_s by $-2q\,\delta_t/3k$, wherein δ_t is the thickness of the thermal boundary layer. Following through the sequence of operations as in the aforementioned reference, one finds

$$T_w - T_\infty = \frac{C}{Re_x^{1/2} Pr^{1/3}} \frac{qx}{k} \left[1 - \frac{x_o}{x}\right]^{1/3}, \; x > x_o \tag{12}$$

The constant C which follows from the integral-method analysis is 2·39. However, we will set aside this value and utilize instead the numerical values from the exact, previously obtained solutions of the boundary-layer energy equation for uniform wall heat flux. In particular, for $x_o = 0$, equation (12) reduces to the dimensionless form

$$\frac{Nu_x}{Re_x^{1/2} Pr^{1/3}} = \frac{1}{C}. \tag{13}$$

It is proposed that the constant C be taken as the reciprocal of the entries in the $n = 0$ row of Table 1. In essence, this represents a matching of the approximate and exact solutions for the uniformly heated plate. The foregoing procedure is entirely justifiable in the interests of obtaining the most accurate results possible. Indeed, the main utility of the solution embodied in equation (12) is in providing the delay factor $[1 - (x_o/x)]^{1/3}$.

The extension of equation (12) to the case of arbitrary distributions of surface heat flux is accomplished by replacing q by $\mathrm{d}q(x_o)$, and then integrating over x_o from 0 to the position x at which T_w is to be determined.

$$T_w - T_\infty = \frac{C}{Re_x^{1/2} Pr^{1/3}} \frac{x}{k} \int_0^x \left(1 - \frac{x_o}{x}\right)^{1/3} \mathrm{d}q(x_o) \tag{14}$$

This integral has to be regarded in the Stieltjes sense in order to accommodate step jumps in q. A more convenient form of the foregoing expression is achieved by an integration by parts, thus

$$T_w - T_\infty = \frac{C/3}{Re_x^{1/2} Pr^{1/3}} \frac{x^{2/3}}{k} \int_0^x \frac{q(x_o)}{(x - x_o)^{2/3}} \mathrm{d}x_o \tag{15}$$

Equation (15) can, in principle, be utilized in computing the distribution of surface temperature for any prescribed distribution of surface heat flux. Indeed, $q(x_o)$ may include step jumps as well as continuous variations; moreover, $q(0)$ need not be zero.

It is of definite interest to subject equation (15) to all possible comparisons with available exact solutions. The purpose of such comparisons would be to establish confidence in the validity of the superposition integral, especially since the delay factor $[1 - (x_o/x)]^{1/3}$ which lies at the heart of the superposition was derived from an energy-integral solution. Exact solutions for power-law distributions in the form of equation (1) have been determined in the preceding portion of this paper. These solutions will be employed in investigating the validity of equation (15).

Upon introducing the heat flux distribution $q = bx^n$ into equation (15) and integrating, one finds

$$\frac{Nu_x}{Re_x^{1/2} Pr^{1/3}} = 3/C\beta(n + 1, \tfrac{1}{3}) \qquad (16)$$

where

$$\beta(n + 1, \tfrac{1}{3}) = \frac{\Gamma(n + 1)\Gamma(\frac{1}{3})}{\Gamma(n + \frac{4}{3})}. \qquad (17)$$

The β and Γ respectively represent the β and Γ functions; numerical values of the latter are available in various handbooks and tabulations. It should be emphasized that the n appearing in equations (16) and (17) need not be an integer.

As previously noted, C is to be chosen to bring about agreement of the integral and exact solutions for $n = 0$. Corresponding to this choice, the values of $Nu_x/Re_x^{1/2}Pr^{1/3}$ as computed from equation (16) for $Pr = 1$ are 0·61196, 0·70562, 0·85937, 0·96760 and 1·1315 respectively for $n = 1, 2, 4, 6$ and 10. Upon comparing these with the exact solutions listed in Table 1, it is seen that excellent agreement prevails. Indeed, the deviations in any case are no more than 1·5 per cent. A similar level of accuracy is found to exist for the other Prandtl numbers. The foregoing comparisons offer strong support of the accuracy of the superposition solution (15).

Arbitrary heat flux distribution, turbulent

The turbulent thermal boundary layer does not lend itself readily to solution for a power-law heat flux distribution. However, the case of an arbitrary heat flux distribution can be treated by superposition, provided that the solution is available for a step change in heat flux applied downstream of the hydrodynamic leading edge. The required step-function solution can be derived by employing the usual semi-empirical methods of turbulent flow. Fortunately, this rather tedious task can be circumvented by making use of a theorem recently stated by Hanna and Meyers [10]. These investigators have shown that if the delay factor corresponding to a step-change in surface temperature is $[1 - (x_o/x)^t]^j$, then the delay factor for a step-change in surface heat flux is $[1 - (x_o/x)]^j$. This theorem holds if the form of the temperature profiles is the same for the two problems. For the turbulent boundary layer, Reynolds *et al.* [11] have verified that the Seban–Scesa [8] delay factor, $[1 - (x_o/x)^{9/10}]^{1/9}$, leads to results in good agreement with experiment for the prescribed temperature case. The corresponding delay factor for the prescribed heat flux case is

$$[1 - (x_o/x)]^{1/9}. \qquad (18)$$

With this and with the correlation* of reference [11], the surface temperature distribution corresponding to a step-change in heat flux applied at x_o is

$$T_w - T_\infty = \frac{qx/k}{0{\cdot}0296 Re_x^{4/5} Pr^{3/5}} \left[1 - \frac{x_o}{x}\right]^{1/9}, \quad x > x_o. \qquad (19)$$

This equation is intended to apply for constant-property, non-dissipative flow.

The generalization of equation (19) to apply to cases of arbitrarily variable surface heat flux is carried out along lines similar to those described for the laminar case. The end result of these operations is

$$T_w - T_\infty = \frac{x^{8/9}/9k}{0{\cdot}0296 Re_x^{4/5} Pr^{3/5}} \int_0^x \frac{q(x_o)\, \mathrm{d}x_o}{(x - x_o)^{8/9}} \qquad (20)$$

* The actual correlation contains a factor $(T_w/T_\infty)^{0{\cdot}4}$ to account for variable gas properties. In the present investigation, this factor will be omitted in favor of a reference temperature for evaluating the properties.

Aerodynamic heating

In applications involving high-speed flow, there may be appreciable dissipation of mechanical energy in the boundary layer. In accounting for this effect, it is customary to solve first for the so-called adiabatic wall temperature T_{aw}. This is the temperature that the surface would achieve if the convective heat transfer were locally zero. For the flow of gases over a flat plate, it is customary to write

$$T_{aw} - T_\infty = RU_\infty^2/2c_p \qquad (21)$$

in which $R = Pr^{1/2}$, laminar flow (21a)

$R = Pr^{1/3}$, turbulent flow. (21b)

Then, with the adiabatic wall temperature in hand, one may generalize equations (15) and (20) to apply to flows with aerodynamic heating. Indeed, it is only necessary to replace T_∞ by T_{aw} in the aforementioned equations. The fluid properties in these equations are to be evaluated at the reference temperature, e.g. Eckert [12].

SIMULTANEOUS CONVECTIVE AND RADIATIVE HEAT TRANSFER

Consideration is now given to a flat plate that exchanges heat by convection with a flowing fluid and by radiation with the surrounding environment. The fluid is assumed to be transparent to radiation and is therefore necessarily a gas.

The governing equations and their solution

Suppose that radiation arriving per unit time and area at the plate surface from various sources in the environment is described by quantities $e_{r1}, e_{r2}, \ldots, e_{rn}$. In general, the absorptance α of the surface for each of these radiation quantities may be different (especially if the spectral distributions are different). If e_{RAD} denotes the rate at which radiant energy is locally absorbed per unit time and area, then

$$e_{RAD} = \sum \alpha_i e_{ri}. \qquad (22)$$

The local emission of the plate surface per unit time and area is $\epsilon\sigma T_w^4(x)$, where ϵ is the emittance. In addition, for the sake of generality, one may suppose that there is an energy flux e_p that must be transferred from the plate to the fluid and to the environment (e.g. an internal heat source or heat sink). In general, both e_{RAD} and e_p may vary with x.

In order that energy conservation be satisfied at each point on the plate surface, it follows that the convective heat transfer q per unit time and area is given by

$$q(x) = e_{RAD}(x) - \epsilon\sigma T_w^4(x) + e_p(x). \qquad (23)$$

Equation (23) is a requisite for the preservation of steady-state conditions. In the development that follows, it will be assumed that $(e_{RAD} + e_p)$ is a known constant, to be denoted by e and referred to as the total heat load. With this, the foregoing becomes

$$q = e - \epsilon\sigma T_w^4(x). \qquad (23a)$$

It may be noted that there is no essential difficulty in treating the case $e = e(x)$. However, it is natural to begin with $e =$ constant; also, this facilitates comparison with the analysis of Cess [2].

Attention may now be given to determining the surface temperature corresponding to the heat flux distribution of equation (23a). For laminar flow with aerodynamic heating, one replaces T_∞ in equation (15) with T_{aw} and arrives at

$$T_w - T_{aw} = \frac{C/3}{Re_x^{1/2}Pr^{1/3}}\frac{x^{2/3}}{k}\int_0^x \frac{q(x_o)}{(x-x_o)^{2/3}}\,\mathrm{d}x_o \qquad (24)$$

Then, upon substitution of equation (23a) and integration, there follows after non-dimensional rearrangement

$$\theta(X) = 1 + X^{1/2}\left(\frac{e}{\epsilon\sigma T_{aw}^4}\right) - \frac{X^{1/6}}{3}\int_0^X \frac{\theta^4(\xi)}{(X-\xi)^{2/3}}\,\mathrm{d}\xi \qquad (25)$$

wherein $\theta = T_w/T_{aw}$, $X = (h_{RAD}/h_{UHF})^2$. (26)

The θ variable compares the wall temperature at x with the adiabatic wall temperature due to convective transport, equation (21). The X variable is readily interpreted by displaying h_{RAD} and h_{UHF}

$$h_{RAD} = \epsilon\sigma T_{aw}^3,\quad h_{UHF} = (k/Cx)Re_x^{1/2}Pr^{1/3}. \qquad (27)$$

From these, it is seen that X is proportional to x. The h_{RAD} is a general measure of the strength of the surface radiation and has the units of a heat-transfer coefficient. Inasmuch as T_{aw} is independent of x, so is h_{RAD}. The h_{UHF} is the local heat-transfer coefficient for laminar flow over a plate with uniform heat flux; clearly, $h_{UHF} \sim x^{-1/2}$. Therefore, h_{RAD}/h_{UHF} is a measure of the relative strength of the surface radiation and the convection, and X has a corresponding meaning. At the leading edge, $h_{UHF} \to \infty$ and $h_{RAD}/h_{UHF} = 0$. With increasing downstream distances, h_{RAD}/h_{UHF} increases monotonically.

The parameter $e/\epsilon\sigma T_{aw}^4$ compares the external heat load with the rate at which energy is radiated from a plate surface at temperature T_{aw}.

Consideration may now be given to solving equation (25), which constitutes a non-linear integral equation for the wall temperature distribution $\theta(X)$. An exact closed-form solution cannot be found; however, highly-accurate numerical solutions can be carried out without difficulty. For this purpose, one divides the region between $X = 0$ and $X = X_{\max}$ into N intervals $\Delta X = X_{\max}/N$. $X_{\max}$ represents the largest X value at which results are desired. Any X value in the range of $0 \leqslant X \leqslant X_{\max}$ is characterized by $X = j\Delta X$; additionally, $\xi = i\Delta X$. The interval ΔX is chosen sufficiently small to insure that within a desired accuracy

$$\int_{i\Delta X}^{(i+1)\Delta X} \frac{\theta^4(\xi)}{(X-\xi)^{2/3}}\,\mathrm{d}\xi = \overline{\theta_i^4} \int_{i\Delta X}^{(i+1)\Delta X} \frac{\mathrm{d}\xi}{(X-\xi)^{2/3}} \tag{28}$$

in which

$$\overline{\theta_i^4} = \tfrac{1}{2}(\theta_i^4 + \theta_{i+1}^4). \tag{29}$$

With this, equation (25) becomes

$$\theta_j = 1 + X^{1/2}\left\{\frac{e}{\epsilon\sigma T_{aw}^4} + j^{-1/3}\sum_{i=1}^{j} \overline{\theta_{i-1}^4}\,[(j-i)^{1/3} - (j-i+1)^{1/3}]\right\}. \tag{30}$$

In applying equation (30), one uses a predictor–corrector technique as described below. First of all, it may be noted that $\theta(0) = 1$, and this is taken as the starting point of the computation. Proceeding to the next point $X = 1 \times \Delta X$ (i.e. $j = 1$), one writes

$$\theta_1 = 1 + (\Delta X)^{1/2}\left\{\frac{e}{\epsilon\sigma T_{aw}^4} - \overline{\theta_0^4}\right\} \tag{31}$$

$\overline{\theta_0^4}$ is first evaluated as $\theta_0^4 = 1$, and a value of θ_1 is then computed. With this, one re-evaluates $\overline{\theta_0^4}$ in accordance with the averaging indicated in equation (29), and this leads to a corrected value of θ_1. A second correction for θ_1 might be carried out, but this was not necessary in view of the small step size employed. Then, passing on to the point $X = 2 \times \Delta X$, one re-applies equation (30)

$$\theta_2 = 1 + (2\Delta X)^{1/2}\left\{\frac{e}{\epsilon\sigma T_{aw}^4} + (2)^{-1/3}\,[\overline{\theta_0^4}(1 - 2^{1/3}) - \overline{\theta_1^4}]\right\}. \tag{32}$$

Inasmuch as θ_0 and θ_1 are known, $\overline{\theta_0^4}$ can be computed in accordance with equation (29); $\overline{\theta_1^4}$ is first evaluated as θ_1^4. With these, a tentative value of θ_2 is computed from equation (32). This is then utilized in conjunction with θ_1 to re-evaluate $\overline{\theta_1^4}$, and in turn, this permits the computation of a refined value of θ_2. The determination of θ_3, θ_4, ... proceeds in a similar manner.

From the foregoing description and from equations (31) and (32), it is evident that the computation procedure makes repetitive use of the third root of integers. Consequently, to minimize the use of computer time, the third roots were determined at the very beginning of the computation and then stored. In addition, it was found possible to increase the step size at larger values of X without loss of accuracy.

Preliminary solutions for θ as a function of X were carried out for various step sizes. By study of these results, one could select step sizes for the final runs that would insure very high accuracy. Indeed, in all cases, *the final results are believed accurate to at least one and perhaps two significant figures beyond those shown in the forthcoming plots*. The presentation and discussion of results will be delayed until after the computation procedure for the turbulent case has been outlined.

For turbulent boundary-layer flow, one proceeds as in the foregoing, except that the basic equations are now (20), (21) and (21b). Into these, one introduces the convective heat flux as indicated by equation (23a). The resulting integral equation for T_w can be rendered dimensionless by defining a coordinate $X = (h_{RAD}/h_{UHF})^5$ and a temperature variable $\theta = T_w/T_{aw}$. Since $h_{RAD}/h_{UHF} \sim x^{1/5}$ for the turbulent case, it follows that such an X is proportional to x. However, this choice for the X coordinate is impractical inasmuch as an exceedingly small step size ΔX would be required for computations at small X. Instead, one defines

$$\chi = h_{RAD}/h_{UHF} \tag{33}$$

wherein h_{RAD} remains as stated in equation (27) while h_{UHF} is the heat-transfer coefficient for turbulent flow over a plate with uniform heat flux

$$h_{UHF} = 0{\cdot}0296(k/x)Re_x^{4/5}Pr^{3/5} \tag{34}$$

It is evident that $\chi \sim x^{1/5}$.

The governing integral equation is reduced to a tractable form by a procedure identical to that described for the laminar case. If $\chi = j\Delta\chi$ and $\xi = i\Delta\chi$, the algebraic approximation to the integral equation for the turbulent case is

$$\theta j = 1 + \chi\left\{\frac{e}{\epsilon\sigma T_{aw}^4} + j^{-5/9}\sum_{i=1}^{j}\overline{\theta_{i-1}^4}\right.$$
$$\left.[(j^5 - i^5)^{1/9} - \{j^5 - (i-1)^5\}^{1/9}]\right\}. \tag{35}$$

The numerical treatment of this equation is similar to that discussed in relation to equation (30).

Distribution of surface temperature

Results for the surface temperature distribution have been computed as described in the foregoing and are presented as solid lines in Figs. 1 and 2. The first of these corresponds to laminar flow and the second to turbulent flow. The ordinate variable is the ratio of the local surface temperature at position x to the adiabatic wall temperature due to convection alone, equation (21). The abscissa for the laminar case is proportional to $x^{1/2}$, while that for the turbulent case is proportional to $x^{1/5}$. Further, the abscissa can be interpreted as a ratio of radiative to convective heat-transfer coefficients, as is noted on the figures. The constant C in the abscissa variable of Fig. 1 is the reciprocal of the $n = 0$ entries of Table 1. The curves are parameterized by $e/\epsilon\sigma T_{aw}^4$. In addition to the solid lines, other curves are presented in the figures. These will be discussed later.

An overall inspection of the figures reveals that the surface temperature may either increase or decrease along the plate depending on the magnitude of $e/\epsilon\sigma T_{aw}^4$. For values of this parameter greater than unity, the temperature increases; the opposite variation occurs when the parameter is less than unity. For the case of $e/\epsilon\sigma T_{aw}^4 = 1$, it is seen that $T_w = T_{aw}$ at all locations.

These trends may be made plausible by physical reasoning. In the absence of a heat load (input) e, there is an energy loss from the surface due to radiation. This heat must be supplied to the plate surface by convection, i.e. $q \sim (T_w - T_{aw}) < 0$. Therefore, $T_w < T_{aw}$. Inasmuch as the convective coefficient decreases with x at a faster rate than does q, it follows that the difference between T_w and T_{aw} increases with increasing x. For moderate heat loads e, the heat loss due to radiation still requires a convective heat transfer into the plate in order to satisfy the local energy balance, equation (23a). Correspondingly, $T_w < T_{aw}$. When $e/\epsilon\sigma T_{aw}^4 = 1$, the heat load is precisely in balance with the surface radiation and the convection is not called upon to transfer heat. For heating conditions such that $e/\epsilon\sigma T_{aw}^4 > 1$, the surface temperature must rise above T_{aw} in order that convection and surface radiation may work together to dissipate the larger heat load.

At the far right of each figure is an array of horizontal line segments that are labelled asymptotes. These apply to the condition $h_{RAD}/h_{UHF} \rightarrow \infty$. This implies that the convective heat transfer q approaches zero, so that the heat balance (23a) becomes $e = \epsilon\sigma T_w^4$. From this, it follows that

$$\frac{T_w}{T_{aw}} = \left(\frac{e}{\epsilon\sigma T_{aw}^4}\right)^{1/4} \tag{36}$$

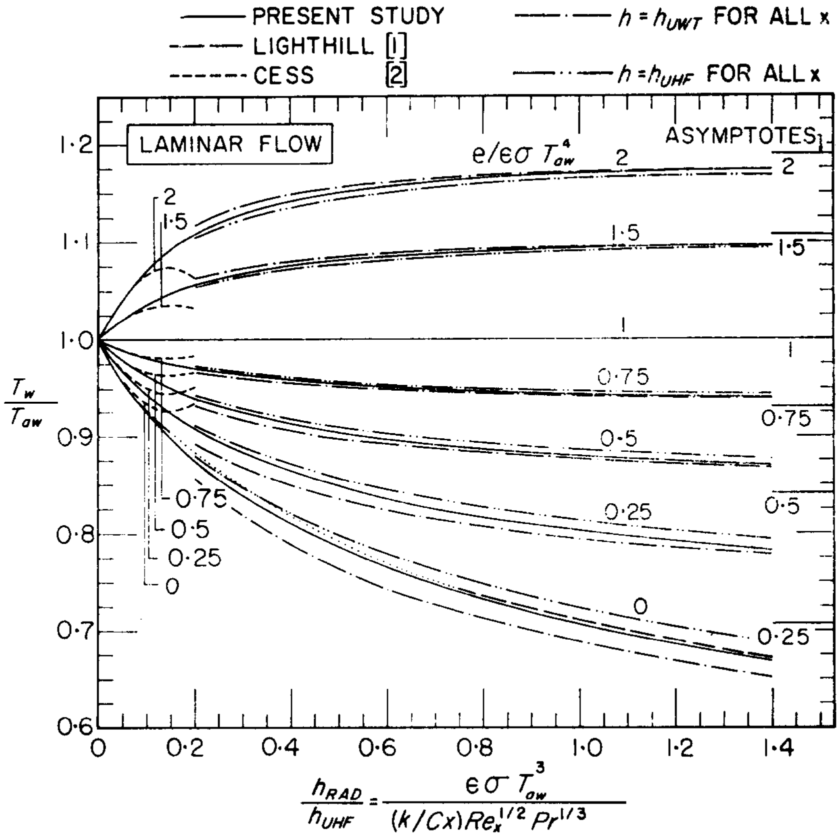

FIG. 1. Surface temperature distribution for laminar flow over a flat plate.

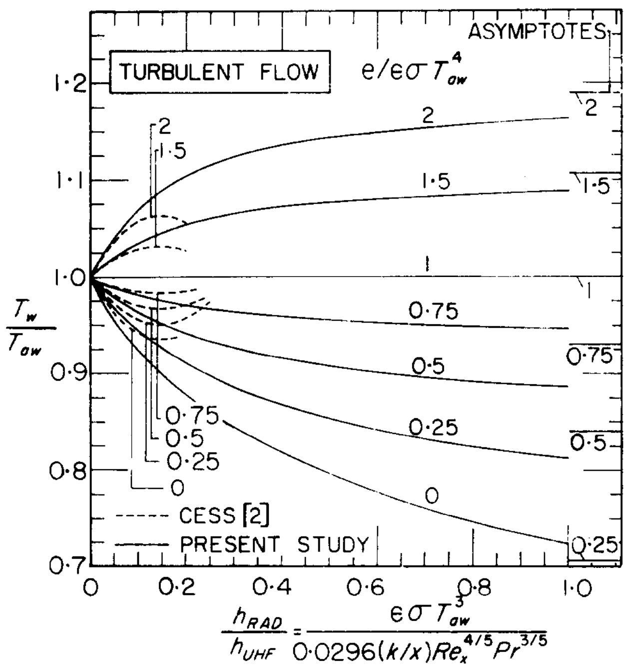

FIG. 2. Surface temperature distribution for turbulent flow over a flat plate.

By inspection of the figures, it is seen that the asymptotes are approached more rapidly for cases in which $e/\epsilon\sigma T_{aw}^4$ is near unity.

In addition to the solutions of the integral equations (solid lines, Figs. 1 and 2), auxiliary approximate calculations have also been carried out. These are based on the local energy balance, equation (23a), into which has been substituted $q = h^*(T_w - T_{aw})$. The convective coefficient h^* will, in general, depend on the specific distribution of surface temperature or of surface heat flux; consequently, h^* will not be known *a priori*. Upon rearrangement, one finds

$$\frac{T_w(x)}{T_{aw}} = \left(\frac{e}{\epsilon\sigma T_{aw}^4}\right)\frac{h_{RAD}}{h^*} - \left[\frac{T_w(x)}{T_{aw}}\right]^4\frac{h_{RAD}}{h^*} + 1. \tag{37}$$

It is natural to seek solutions for the surface temperature distribution corresponding to heat-transfer coefficients for the two standard cases of uniform heat flux (*UHF*) and uniform wall temperature (*UWT*). Such solutions have been carried out and are plotted in Fig. 1 as dash–dot (*UWT*) and dash–double dot (*UHF*) lines. It is seen that, in general, the results based on h_{UHF} and h_{UWT} bracket the exact solutions (solid lines) for the surface temperature distribution. Indeed, the bracketing curves form rather close bounds on the exact solution. For $e/\epsilon\sigma T_{aw}^4 < 1$, the computation based on h_{UHF} overestimates the local surface temperature while that based on h_{UWT} underestimates the local surface temperature. For $e/\epsilon\sigma T_{aw}^4 > 1$, opposite findings apply.

For the turbulent case, the heat-transfer coefficients for uniform heat flux and uniform wall temperature are essentially identical. When equation (37) is solved for the condition $h^* = h_{UHF} = h_{UWT}$ and the results plotted in Fig. 2, it is found that the points fall very close to the solid curves that represent the solution of the integral equation (35). Indeed, any differences would be obscured during the preparation of ink drawings. This finding reaffirms the well-established physical concept that a turbulent boundary layer has a "poor memory" relative to the details of the upstream thermal boundary conditions. It would thus appear that any additional solutions for the convective–radiative heat-transfer problem under turbulent conditions could best be carried out with the algebraic equation (37) instead of the integral equation (35).

Figures 1 and 2 also contain curves which represent the results of prior investigations by Lighthill and by Cess. The former limited his considerations to the laminar case with $e = 0$, i.e. radiative cooling of an aerodynamically heated plate. Series solutions for small and large x are shown as long-dashed lines; these are connected by a dotted curve that corresponds to an interpolation. In general, Lighthill's results lie slightly higher than those of the present analysis; but, the agreement is quite satisfactory.

Cess extended consideration to both laminar and turbulent flows. Aerodynamic heating was neglected, but the analysis is readily generalized to include these effects. Cess' results, originally reported as Nusselt numbers, have been rephrased in terms of the temperature ratio T_w/T_{aw} and are plotted in Figs. 1 and 2 as short-dashed lines. It is seen from the figures that these are valid only in the range of small radiative effects; this is precisely the condition for which the Cess analysis was carried out.

CONCLUDING REMARKS

It may be of interest to point out the particular advantage of using the prescribed-heat-flux formulation in solving problems of the type considered here. The essential point is that one deals with linear differences in position $[1 - (x_o/x)]^j$, for instance, equations (12) and (19). Thus, after applying the mean value theorem in equation (28), the integration results in a simple function. On the other hand, had the prescribed-wall-temperature formulation been employed, much more complicated functions would have been obtained at this point in the analysis. This is because in the latter formulation, the differences in position are non-linear, that is $[1 - (x_o/x)^i]^j$.

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Résumé—Une analyse est effectuée pour déterminer la distribution de température pariétale le long d'une plaque plane soumise simultanément à un transport de chaleur par convection, à un échange par rayonnement avec l'ambiance, à un échauffement aérodynamique, et à des sources ou des puits de chaleur internes. On considère à la fois des écoulements de couche limite laminaire et turbulente. Des résultats numériques sont présentés pour une large gamme des paramètres influents; ceux-ci sont comparés avec des solutions simplifiées basées sur l'application locale des coefficients de transport de chaleur pour une température pariétale uniforme et pour un flux de chaleur uniforme.

Le problème est traité dans le cadre de la couche limite thermique avec un flux de chaleur imposé. La partie initiale de l'article est consacrée à l'établissement de certains résultats généraux pour de telles couches limites. Des solutions exactes sont obtenues pour des distributions de flux de chaleur en puissance et sous forme de séries. Une solution approchée pour un flux de chaleur pariétal variant arbitrairement est obtenue en superposant des solutions en marche d'escalier fournies par l'équation intégrale de l'énergie.

Zusammenfassung—Um die Temperaturverteilung an der Oberfläche einer ebenen Platte zu bestimmen, die gleichzeitig Wärmeübergang durch Konvektion, Strahlungsaustausch mit der Umgebung und aerodynamische Erwärmung erfährt und bei welcher innere Wärmequellen und -senken auftreten, wird eine Analyse ausgeführt. Es werden sowohl die laminaren wie auch die turbulenten Grenzschichtströmungen betrachtet. Für einen weiten Bereich der bestimmenden Parameter werden numerische Ergebnisse angegeben. Sie werden mit vereinfachten Lösungen verglichen, die auf der lokalen Verwendung der Wärmeübergangszahl bei gleichförmiger Wandtemperatur und bei gleichförmiger Wärmestromdichte beruhen.

Das Problem wird innerhalb der Rahmenarbeit über die thermische Grenzschicht mit vorgegebener Wärmestromdichte behandelt. Der erste Teil der Abhandlung befasst sich mit der Aufstellung bestimmter allgemeiner Resultate für solche Grenzschichten. Für Verteilungen der Wärmestromdichte nach Potenzgesetzen und -reihen ergaben sich exakte Lösungen. Für willkürlich veränderliche Wärmestromdichten an der Oberfläche wird eine Näherungslösung abgeleitet, indem Lösungen in Form Stufenfunktionen, die sich aus der Integral-Energiegleichung ergeben, superponiert werden.

Аннотация—В данной работе проведен анализ распределения температуры поверхности вдоль плоской пластины при одновременном действии конвективного переноса тепла, лучистого обмена с окружающей средой, аэродинамического нагрева и наличии внутренних источников и стоков тепла. Рассмотрены как ламинарное, так и турбулентное течения в пограничном слое. Представлены численные результаты для широкого диапазона основных параметров и дано их сравнение с упрощенными решениями, полученными на основе локального использования коэффициентов теплообмена для однородной температуры стенки и однородного теплового потока.

Задача рассматривается в рамках теплового пограничного слоя при заданном тепловом потоке. Начальная часть статьи посвящена установлению определенных общих результатов для таких пограничных слоев. Получены точные решения для распределений теплового потока, заданных степенным законом и в виде разложения в ряд. Получено приближенное решение для произвольно изменяющейся величины теплового потока на поверхности путём суперпозиции ступенчатых решений, полученных из интегрального уравнения энергии.